

Natural octonionic generalization of general relativity

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An intriguingly natural generalization, using complex octonions, of general relativity is pointed out. The starting point is the vierbein-based double dual formulation of the Einstein-Hilbert action. In terms of two natural structures on the (complex) quaternions and (complex) octonions, the inner product and the cross products, respectively, this action is linked with the complex quaternionic structure constants, and subsequently generalized to an achtbein-based 'double χ -dual' action in terms of the complex octonionic structure constants.

It is the purpose of this note to point out an intriguingly natural generalization of general relativity, using complex octonionic structures.

In terms of the doubly contracted, double dual [1, p. 325] of the Riemann curvature tensor $R_{\mu\nu}{}^{\rho\sigma}$, the Einstein-Hilbert action may be written as

$$S_{\text{EH}} = \int L_{\text{EH}} \sqrt{-g} d^4x,$$

$$L_{\text{EH}} = -\frac{c^4}{64\pi G} \varepsilon^{\alpha\beta\mu\nu} R_{\mu\nu}{}^{\rho\sigma} \varepsilon_{\rho\sigma\alpha\beta},$$

where $g \equiv \det(g_{\mu\nu})$ is the determinant of the metric, and $\varepsilon_{\mu\nu\rho\sigma}$ and $\varepsilon^{\mu\nu\rho\sigma}$ are the Levi-Civita tensor densities [1, Eq. (8.10)]

$$\varepsilon_{\mu\nu\rho\sigma} = +(-g)^{+1/2} [\mu\nu\rho\sigma],$$

$$\varepsilon^{\mu\nu\rho\sigma} = -(-g)^{-1/2} [\mu\nu\rho\sigma],$$

where $[\mu\nu\rho\sigma]$ is a completely antisymmetric symbol with $[0123] = +1$. This assertion is an immediate consequence of the identity

$$-\frac{1}{2} \varepsilon^{\alpha\beta\mu\nu} \varepsilon_{\alpha\beta\rho\sigma} = \delta_\rho^\mu \delta_\sigma^\nu - \delta_\sigma^\mu \delta_\rho^\nu \equiv \delta_{\rho\sigma}^{\mu\nu},$$

where $\delta_{\rho\sigma}^{\mu\nu}$ is a generalized Kronecker delta [2, Sect. 4.2].

In terms of a vierbein $e^a{}_\mu$, with corresponding minimal spin connection $\omega_\mu{}^{ab} \equiv g^{\rho\sigma} e^a{}_\rho \nabla_\mu e^b{}_\sigma$, the Einstein-Hilbert action may equivalently be written as

$$S = \int L e d^4x,$$

$$L = -\frac{c^4}{64\pi G} \varepsilon^{\alpha\beta\mu\nu} e^\rho{}_a e^\sigma{}_b R_{\mu\nu}{}^{ab} \varepsilon_{\rho\sigma\alpha\beta},$$

where $e \equiv \det(e^a{}_\mu)$ is the determinant of the vierbein, $R_{\mu\nu}{}^{ab}$ is the curvature tensor of the minimal spin connection [3, 4], and $\varepsilon_{\mu\nu\rho\sigma}$ and $\varepsilon^{\mu\nu\rho\sigma}$ are now given by

$$\varepsilon_{\mu\nu\rho\sigma} = e^a{}_\mu e^b{}_\nu e^c{}_\rho e^d{}_\sigma \varepsilon_{abcd},$$

$$\varepsilon^{\mu\nu\rho\sigma} = e^\mu{}_a e^\nu{}_b e^\rho{}_c e^\sigma{}_d \varepsilon^{abcd},$$

because $\varepsilon_{abcd} = -\varepsilon^{abcd} = [abcd]$ in any Minkowski frame.

By itself, all of the above would of course be quite pointless, were it not for the following fact: The above

vierbein-based double dual action allows for an intriguingly natural (i.e., uncontrived) generalization, in terms of complex octonions, of general relativity, using the complex quaternions as a stepping-stone. In order to analytically formulate this assertion, two structures, the inner product and the triple cross products, see shortly, must first be defined.

Remark: In this note, no formal introduction to the (complex) quaternions or (complex) octonions will be given. Instead, the reader is kindly referred to the literature: For short reviews of the octonions, see for instance Refs. [5, 6, 7, 8]. For a comprehensive review of the octonions, see Ref. [9]. For a monograph on octonions and other nonassociative algebras, see Ref. [10]. In particular, for a monograph on composition algebras, a class to which both the complex quaternions and complex octonions belong (note that they are not division algebras, even though the quaternions and the octonions themselves are), see Ref. [11]. For material on the quaternions, see for instance Refs. [9, 10, 12].

Below, \mathbb{D} (for division algebra) denotes either the set of quaternions \mathbb{H} , or the set of octonions \mathbb{O} . Standardly, although it differs in various references by a normalizing factor of 2, define an inner product $\langle \cdot, \cdot \rangle : (\mathbb{C} \otimes \mathbb{D})^2 \rightarrow \mathbb{C}$ by

$$2 \langle x, y \rangle = x\bar{y} + y\bar{x} \equiv \bar{x}y + \bar{y}x,$$

where \bar{x} denotes the (quaternionic or octonionic) conjugate of x . Define triple cross products $X_L, X_R : (\mathbb{C} \otimes \mathbb{D})^3 \rightarrow \mathbb{C} \otimes \mathbb{D}$ by

$$3!X_L(x, y, z) = x(\bar{y}z - \bar{z}y) + \text{cyclic perm},$$

$$3!X_R(x, y, z) = (x\bar{y} - y\bar{x})z + \text{cyclic perm},$$

where L and R means left and right, respectively. The cross products X_L and X_R possess both the orthogonality property and the (generalized) Pythagorean property [13];

$$0 = \langle X(x_1, x_2, x_3), x_i \rangle,$$

$$\det(\langle x_i, x_j \rangle) = \langle X(x_1, x_2, x_3), X(x_1, x_2, x_3) \rangle,$$

where the suppressed subscript means that the relations apply to both L and R . Trilinear cross products possessing both these properties exist only over algebras of

real (or complex) dimension 4 or 8 [13, 14], the underlying reason being the existence of precisely the division algebras \mathbb{H} and \mathbb{O} .

Remark: For $\mathbb{D} = \mathbb{H}$, the cross products X_L and X_R are trivially identical because the quaternions are associative, so for the quaternions the subscript will be dropped, writing just X (for both). For $\mathbb{D} = \mathbb{O}$, however, this is not the case because the octonions are nonassociative.

As bases (over \mathbb{C}) for $\mathbb{C} \otimes \mathbb{H}$ and $\mathbb{C} \otimes \mathbb{O}$, respectively, define $e_a = (i, e_i)$ and $E_a = (i, E_i)$, where i is the complex imaginary unit, e_i are the standard units of the pure quaternions $\text{Im } \mathbb{H}$ (quaternions with no real part), and E_i are the standard units of the pure octonions $\text{Im } \mathbb{O}$ (octonions with no real part). The units e_i obey

$$\begin{aligned} 2\varepsilon_{ij}{}^k e_k &= [e_i, e_j], \\ 0 &= [e_i, e_j, e_k], \end{aligned}$$

where ε_{ijk} are the completely antisymmetric structure constants of the quaternions (with $\varepsilon_{123} = +1$). The units E_i obey

$$\begin{aligned} 2\psi_{ij}{}^k E_k &= [E_i, E_j], \\ 2\phi_{ijk}{}^l E_l &= [E_i, E_j, E_k], \end{aligned}$$

where $\psi_{ijk} \in \mathbb{R}$ and $\phi_{ijkl} \in \mathbb{R}$ (each others dual in \mathbb{R}^7) are the completely antisymmetric structure constants of the octonions. Above, $[\cdot, \cdot] : (\mathbb{C} \otimes \mathbb{D})^2 \rightarrow \mathbb{C} \otimes \mathbb{D}$ and $[\cdot, \cdot, \cdot] : (\mathbb{C} \otimes \mathbb{D})^3 \rightarrow \mathbb{C} \otimes \mathbb{D}$, defined by

$$\begin{aligned} [x, y] &= xy - yx, \\ [x, y, z] &= (xy)z - x(yz), \end{aligned}$$

are the commutator and associator, respectively.

Remark: The following index conventions are adhered to throughout. Lower case Latin letters from the middle of the alphabet, starting at i , either run from 1 to 3, or from 1 to 7, depending on the context. They are raised and lowered with δ^{ij} and δ_{ij} , respectively. Lower case Latin letters from the beginning of the alphabet either run from 0 to 3, or from 0 to 7, depending on the context. They are raised and lowered with η^{ab} and η_{ab} , respectively. Lower case greek letters either run from 0 to 3, or from 0 to 7, depending on the context. They are raised and lowered with $g^{\mu\nu}$ and $g_{\mu\nu}$, respectively.

Now, the seemingly insignificant but in fact crucial observation is that

$$\varepsilon_{abcd} = i \langle X(e_a, e_b, e_c), e_d \rangle,$$

a relation linking duality in spacetime, as controlled by ε_{abcd} , with two natural structures of the (complex) quaternions, the inner product and the cross product. This relation is the complex quaternionic stepping-stone, previously referred to.

Because the (complex) quaternions can be embedded in the (complex) octonions (in numerous ways),

it seems natural to generalize as follows: Define $\chi_{abcd}^{(L)}$ and $\chi_{abcd}^{(R)}$ (eight-dimensional generalizations of the four-dimensional ε_{abcd}) by

$$\begin{aligned} \chi_{abcd}^{(L)} &= i \langle X_L(E_a, E_b, E_c), E_d \rangle, \\ \chi_{abcd}^{(R)} &= i \langle X_R(E_a, E_b, E_c), E_d \rangle. \end{aligned}$$

Even though not obvious, $\chi_{abcd}^{(L)}$ and $\chi_{abcd}^{(R)}$ are in fact completely antisymmetric. Subsequently, define $\chi_{\mu\nu\rho\sigma}^{(L)}$ and $\chi_{\mu\nu\rho\sigma}^{(R)}$ (eight-dimensional generalizations of the four-dimensional $\varepsilon_{\mu\nu\rho\sigma}$) by

$$\begin{aligned} \chi_{\mu\nu\rho\sigma}^{(L)} &= E^a{}_\mu E^b{}_\nu E^c{}_\rho E^d{}_\sigma \chi_{abcd}^{(L)}, \\ \chi_{\mu\nu\rho\sigma}^{(R)} &= E^a{}_\mu E^b{}_\nu E^c{}_\rho E^d{}_\sigma \chi_{abcd}^{(R)}, \end{aligned}$$

where $E^a{}_\mu$ is an achtbein. Finally, define an achtbein-based 'double χ -dual' action (generalization of the vierbein-based double dual action) by

$$\begin{aligned} S &= \int L E d^8 x, \\ L &= -\frac{c^4}{64\pi G} \chi_{(L)}^{\alpha\beta\mu\nu} E^{\rho}{}_a E^{\sigma}{}_b R_{\mu\nu}{}^{ab} \chi_{\rho\sigma\alpha\beta}^{(R)}, \end{aligned}$$

where $E \equiv \det(E^a{}_\mu)$ is the determinant of the achtbein, and $R_{\mu\nu}{}^{ab}$ is the curvature tensor of the associated minimal spin connection $\Omega_\mu{}^{ab} \equiv g^{\rho\sigma} E^a{}_\rho \nabla_\mu E^b{}_\sigma$. This is the generalized action promised in this note.

By straightforward calculations, the following relations may be obtained, the remaining components following from the complete antisymmetry of both $\chi_{abcd}^{(L)}$ and $\chi_{abcd}^{(R)}$:

$$\begin{aligned} \chi_{0ijk}^{(L)} &= +\chi_{0ijk}^{(R)} = \psi_{ijk}, \\ \chi_{ijkl}^{(L)} &= -\chi_{ijkl}^{(R)} = i\phi_{ijkl}. \end{aligned}$$

These relations show that $\chi_{abcd}^{(L)}$ and $\chi_{abcd}^{(R)}$, and therefore also $\chi_{\mu\nu\rho\sigma}^{(L)}$ and $\chi_{\mu\nu\rho\sigma}^{(R)}$, are each others complex conjugate, implying that the above Lagrangian is real-valued, as befits any reasonable Lagrangian.

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